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An explicit algebraic relation for calculating Reynolds normal stresses in flows dominated by bubble-induced turbulence

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Two new algebraic turbulence models for flows dominated by bubble-induced turbulence (BIT) are presented. They combine different elements of existing models that are considered superior to their alternatives. Both models focus on the core region of a channel flow, where bubble-induced production and dissipation are nearly in balance and the void fraction is approximately homogenous. The first model, referred to as the algebraic Reynolds normal stress model, is derived from the differential Reynolds stress model of Ma *et al.* (*J. Fluid Mech.* 883, A9, 2020). The second model utilizes the original two-equation turbulence model for bubbly flows (Ma *et al.* *Phys. Rev. Fluids* 2, 034301, 2017) to achieve algebraic expressions for k and ε in the BIT dominated cases. If both models are combined, it results in a purely algebraic, explicit relation for the Reynolds normal stresses, that only depends on the mean flow parameters, namely, mean gas void fraction, and mean liquid and gas velocities. We find that the model can well predict the Reynolds normal stresses, when compared with direct numerical simulation and experimental data.

I. INTRODUCTION

For large-scale turbulent bubbly flow simulations, the Euler Euler (EE) approach [1] (see the governing eqs. (A3) and (A4) in Appendix A) coupled with steady or unsteady Reynolds-averaged Navier-Stokes (RANS) modeling is the only viable framework. In this case only continuous statistical quantities are computed, so that beyond the closures for single-phase terms all two-phase phenomena related to the phase boundaries need to be modeled. Except for the prediction of the gas void fraction, the main challenge when using the EE RANS approach for turbulent bubbly flows is in determining the Reynolds stresses appearing in the EE momentum equations.

Over the past two decades, considerable work has been done to develop single phase two-equation linear eddy viscosity models (LEVM) to include specific source terms capturing the effect of bubble-induced turbulence (BIT) [2–7]. These models take the influence of bubbles into account by including additional source terms in the balance equations for both k, the turbulent kinetic energy (TKE), and ε, the turbulent kinetic energy dissipation rate, or another equivalent parameter. This alters the turbulence quantities and, as a result, the effective transport coefficients, such as the eddy viscosity. However, despite the fact that these models are able to predict the TKE and dissipation well, the approach suffers from substantial uncertainties concerning the concept of the LEVM

$$-\overline{u_i'u_j'} = \underbrace{C_\mu \frac{k^2}{\varepsilon}}_{\nu_t}\left(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\right) - \frac{2}{3}\delta_{ij}k \tag{1}$$

when applied to turbulent bubbly flows, as discussed in [8]. In the expression above, $\overline{u}$ is the mean liquid velocity, C_μ is a constant, ν_t is the eddy viscosity, δ_{ij} is the Kronecker delta, and the fluctuation of the velocity is defined as $u' = u - \overline{u}$. Here, $\overline{\cdots}$ denotes the Reynolds averaging with respect to time, space or ensemble of realizations.

For single-phase flows, there is some rational for eddy viscosity closures in that for such a flow, the turbulence production depends essentially upon the mean velocity gradients doing work on the Reynolds stresses in the flow. However, for BIT dominated flows, the turbulence production is mainly associated with the interfacial energy transfer between bubbles and liquid, rather than production by the mean velocity gradients [9]. As a result, closing the Reynolds stresses in terms of the mean fluid velocity gradient may not make sense for BIT dominate flows, and this issue cannot be overcome by trying the better represent the effect of the bubbles in the eddy viscosity itself.

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In view of these challenges, a number of studies have sought to develop Reynolds stress closures that are more suitable for BIT dominated flows, and differential Reynolds stress models (DRSM) based on the EE approach were developed by several groups [8, 10–13]. One particular form of a DRSM, which will later serve as the starting point for deriving the new explicit algebraic Reynolds stress model (EARSM), is given by the linearised model of [8](the rapid part in the pressure strain is simplified as well):

$$\begin{aligned}
&\underbrace{(1-\alpha)\frac{D\overline{u_i'u_j'}}{Dt}}_{C_{ij}:\ convection} - \underbrace{(1-\alpha)\frac{\partial}{\partial x_k}\left((\nu + c_s\nu_t)\frac{\partial \overline{u_i'u_j'}}{\partial x_k}\right)}_{D_{ij}:\ diffusion} = \underbrace{-(1-\alpha)\left(\overline{u_i'u_k'}\frac{\partial \overline{u_j}}{\partial x_k} + \overline{u_j'u_k'}\frac{\partial \overline{u_i}}{\partial x_k}\right)}_{P_{ij}:\ production} \\
&\underbrace{-(1-\alpha)\frac{2}{3}\delta_{ij}\varepsilon}_{\varepsilon_{ij}:\ dissipation} \underbrace{-c_1(1-\alpha)\varepsilon\left(\frac{\overline{u_i'u_j'}}{k} - \frac{2}{3}\delta_{ij}\right) - c_2\left(P_{ij} - \frac{1}{3}\delta_{ij}P_{kk}\right)}_{\phi_{ij}:\ pressure\ strain} + \underbrace{S_{R,ij}}_{interfacial} \ ,
\end{aligned} \tag{2}$$

with

$$S_{R,ij} = \begin{pmatrix} \underbrace{\min(0.67 + 0.67\exp(370 Re_p^{-1.2}), 2)}_{b_{11}^*} & 0 & 0 \\ 0 & \underbrace{\frac{1}{2}(2 - b_{11}^*)}_{b_{22}^*} & 0 \\ 0 & 0 & b_{33}^* = b_{22}^* \end{pmatrix} S_k \ , \tag{3}$$

where the interfacial term for the k-equation is adopted from [14](see Appendix A):

$$S_k = \min(0.18 \cdot Re_p^{0.23}, 1)\boldsymbol{F}_D\boldsymbol{u}_r \ . \tag{4}$$

Here, α is the gas void fraction, ν the liquid molecular kinematic viscosity, $c_s = 1.63$, $c_1 = 1.7$, and $c_2 = 0.6$ are empirical constants. Furthermore, the bubble Reynolds number $Re_p = d_p u_r/\nu$, is based on bubble diameter d_p, $\boldsymbol{u}_r$ is the averaged relative velocity between the bubble and liquid (with u_r the component of $\boldsymbol{u}_r$ in the x direction), and the drag force is $\boldsymbol{F}_D = \frac{3}{4d_p}C_D\alpha|\boldsymbol{u}_r|\boldsymbol{u}_r$. On the left-hand side of eq. (2), we have written $(1-\alpha)$ outside of the derivative, since in the present study we focus on homogeneously distributed bubbly flows, for which $D\alpha/Dt = 0$.

In practice, the vast majority of computations for bubbly flows using the EE framework have been made using two-equation models, and DRSMs have not received as enthusiastically as two-equation models by the user community, which is in line with the historical development in the single-phase turbulence models [15]. The reason, of course, is that DRSMs are more computationally demanding than simple two-equation models. However, if one uses a two-equation model, then one does not have direct access to information on the Reynolds stresses. One choice is to estimate them using (1), however, as discussed earlier, this is likely a poor approximation for BIT dominated flows. Another standard approach is to assume isotropy $\overline{u_i'u_j'} \approx \frac{2}{3}\delta_{ij}k$ [12, 16]. However, the buoyancy of the bubbles generally introduces strong anisotropy into the flow, with the Reynolds stress in the direction of gravity being dominant [8, 17, 18]. Therefore, the isotropic approximation can be very inaccurate.

An intermediate modeling strategy between the level of LEVMs and DRSMs are algebraic Reynolds stress models (ARSMs) [19]. In single-phase flows, ARSMs can be obtained by applying a weak-equilibrium hypothesis (discussed later) to a DRSM, such that ARSMs are obtained directly from DRSMs [20]. While this has proven in appropriate contexts to be a fruitful approach in single-phase modeling, surprisingly, there do not seem to be any ARSMs derived based on the available DRSMs for BIT flows. (Masood et al. [21] applied the ARSM model of [22], developed for single-phase flow, to bubbly flows but did not account for any BIT dominated flow effects.)

The main purpose of the present paper is to extend the idea of Rodi [20] to BIT dominated flows, starting from a recently developed DRSM for BIT flows. We show that for BIT dominated flows there is a considerable difference in how ARSM can be derived as compared with the original work by Rodi for single-phase flow. Furthermore, the expression of the new ARSM is explicit in terms of k, ε, and other parameters, in contrast to single-phase flows where ARSM have implicit dependencies, in the case where they are derived directly from DRSMs. We also show that for BIT dominated flows, k and ε can be determined from a purely algebraic relation. This leads to a simplified but accurate modeling strategy for BIT dominated flows that circumvents the need to solve differential equations.

II. DERIVATION OF EXPLICIT ALGEBRAIC MODELS IN BIT DOMINATED CASES

A. Explicit ARSM expression in the BIT dominated cases

The basic idea behind ARSM is to reduce the set of DRSM equations to a system of coupled algebraic equations by approximating the convection and diffusion terms appearing in the DRSM. The simplest approach would be to simply neglect these transport terms, something that is strictly only applicable when the flow is in local equilibrium. A more general approximation in single-phase flows is to apply the "weak non-equilibrium hypothesis" introduced by Rodi [20]. This hypothesis is that the non-dimensional Reynolds-stress anisotropy tensor

$$a_{ij} = \frac{\overline{u_i'u_j'}}{k} - \frac{2}{3}\delta_{ij}, \tag{5}$$

either does not vary (i.e. $Da_{ij}/Dt = 0$), or varies much more slowly than the TKE across the flow field, which leads to

$$\frac{D\overline{u_i'u_j'}}{Dt} = \frac{\overline{u_i'u_j'}}{k}\frac{Dk}{Dt} + k\frac{D\left(\overline{u_i'u_j'}/k\right)}{Dt} \approx \frac{\overline{u_i'u_j'}}{k}\frac{Dk}{Dt}, \tag{6}$$

for the mean convection term, and

$$D_{ij} \approx \frac{\overline{u_i'u_j'}}{k}D_k, \tag{7}$$

for the diffusion term, where $D_k = (1/2)D_{ii}$. In other words, spatial and temporal variations in $\overline{u_i'u_j'}$ are considered to be due to variations in k, while variations in $\overline{u_i'u_j'}/k$ are neglected. For the latter, D_{ij} is proportional to the diffusion of k scaled with the relative intensity in the respective direction.

When we apply the "weak non-equilibrium hypothesis" to bubbly turbulent flows (which amounts to inserting (6) and (7) into (2)), the following algebraic expression for $\overline{u_i'u_j'}$ is obtained

$$\overline{u_i'u_j'} = \frac{k\left(P_{ij} + S_{R,ij} + \frac{2}{3}\delta_{ij}(1-\alpha)(c_1-1)\varepsilon - c_2\left(P_{ij} - \frac{2}{3}\delta_{ij}P_k\right)\right)}{P_k + S_k - (1-\alpha)\varepsilon + c_1(1-\alpha)\varepsilon}\ , \tag{8}$$

where $P_k \equiv \frac{1}{2}P_{ii}$. Similar to the result in Rodi's original single-phase work, this constitutes an implicit algebraic equation for $\overline{u_i'u_j'}$.

Two additional simplifications may be made to the result in (8). First, in BIT dominated flows the interfacial term is the main source of production, with $|P_{ij}| \ll |S_{R,ij}|$ [23], and therefore (8) may be simplified to

$$\overline{u_i'u_j'} = \frac{k\left(S_{R,ij} + \frac{2}{3}\delta_{ij}(1-\alpha)(c_1-1)\varepsilon\right)}{S_k - (1-\alpha)\varepsilon + c_1(1-\alpha)\varepsilon}\ , \tag{9}$$

which is an explicit algebraic equation for $\overline{u_i'u_j'}$. Second, for BIT dominated flows far from the wall, it was shown in [8] using DNS data that local equilibrium is a good approximation. For local equilibrium, the Reynolds stress convection and diffusion terms can be neglected entirely, and (9) further simplifies to

$$\overline{u_i'u_j'} = \frac{k}{c_1(1-\alpha)\varepsilon}S_{R,ij} + \frac{2}{3}\delta_{ij}k\left(1 - \frac{1}{c_1}\right)\ . \tag{10}$$

From this it is seen that the model predicts that the Reynolds stress anisotropy is directly related to the interfacial anisotropy. Moreover, it should be noted that (2), from which (10) is derived, involves modeling assumptions about the dissipation and pressure-strain terms. The relatively simple form of (10) is in part due to the simple Rotta linear return-to-isotropy model for ϕ_{ij} [24], with constant c_1.

The result in (9) is more general and powerful than that in (10) since the former only assumes weak non-equilibrium, while the latter exact local equilibrium. However, without assuming local equilibrium, it is not possible to derive an explicit algebraic equation for the TKE in BIT dominated flows (this result is discussed below), and the derivation of a fully explicit algebraic model is one of the goals of this paper. Therefore, in what follows we use (10) and focus on the region of BIT dominated flows where local equilibrium is a reasonable approximation. Extensions to flows where local equilibrium does not hold, and where (9) could provide a suitable model will be considered in future work.

Parameter	*SmMany*	*LaMany*
N_p	2880	913
α	2.14%	2.14%
d_p/H	0.052	0.076
Ar	38171	114528
Re_p	235.5	475.2

TABLE I: Parameters of the cases used for the present study according to [23]. The labels *Sm* (smaller) and *La* (larger) designate different bubble sizes. Here, N_p is the number of bubbles, α the void fraction, d_p the bubble diameter, Ar the Archimedes number. The values of Re_p are the results of the simulations.

Since the trace of (10) yields $2k = 2k$, eq. (10) does not provide a means for obtaining the TKE, just as is the case for single-phase ARSM. Therefore k, as well as ε, must be determined by other means, for instance from a two equation model for bubbly flows such as the model proposed by Ma et al. [14]. Furthermore, since P_{ij} does not appear in (10), owing to the restriction to BIT dominated flows, (10) is an explicit equation for $\overline{u_i' u_j'}$, unlike the single-phase ARSM of [20] where a system of algebraic equations must be solved to obtain $\overline{u_i' u_j'}$.

B. Explicit algebraic expressions for k and ε in the BIT dominated cases

The other major difference between the present study and the original work of [20], is that for our case simple algebraic expressions may be constructed for k and ε that apply to BIT dominated flows. Taking half the trace of (10) and rearranging yields the expression

$$\varepsilon = \frac{S_k}{(1-\alpha)} \ . \tag{11}$$

Moreover, under the same conditions the model transport equation for ε from [14] (see eq. (A2)) reduces to

$$0 \approx \varepsilon_\varepsilon + S_\varepsilon \ , \tag{12}$$

where the dissipation term is $\varepsilon_\varepsilon = -(1-\alpha)C_{\varepsilon 2}(\varepsilon^2/k)$, the source term is $S_\varepsilon = 0.3 C_D (S_k/\tau)$, $C_{\varepsilon 2} = 1.92$, and $\tau = d_p/u_r$ is the time scale characterizing the BIT dominated flow. Incorporating these expressions into (12) yields the desired algebraic expression for k

$$k = \frac{C_{\varepsilon 2} d_p S_k}{0.3(1-\alpha) C_D u_r} \ . \tag{13}$$

III. COMPARISON OF PREDICTIONS WITH DNS AND EXPERIMENT

Equation (10) together with the explicit algebraic $k-\varepsilon$ models given by (11) and (13) provide a very simple, purely algebraic model for predicting $\overline{u_i' u_j'}$. We now test these algebraic models using DNS and experimental data.

To assess (11) and (13), two test cases (*SmMany* and *LaMany*) of [23] are considered to demonstrate the results for ε and k, respectively. This data is generated from bubble-resolving DNS with many thousands of spherical bubbles at low Eötvös number. The DNS were conducted for upward vertical flow between two flat walls in a channel, with x the streamwise, y the wall-normal, and z the spanwise coordinate. The size of the computational domain is $L_x \times L_y \times L_z = 4.41H \times H \times 2.21H$, where H is the distance between the walls. A no-slip condition was applied at the walls and periodic conditions in x and z. Gravity acts in the negative x-direction, and the bulk velocity U_b was kept constant by instantaneously adjusting a volume force, equivalent to a pressure gradient, thus imposing a desired bulk Reynolds number $Re_b = U_b H/\nu$, where ν is the kinematic viscosity of the liquid. The DNS were all conducted with $Re_b = 5263$. The data used in this work were obtained for two monodisperse cases (*SmMany* and *LaMany*). Table I provides an overview of both cases with the corresponding labels. The data available cover statistical moments of first and second order for liquid and bubbles.

The results were evaluated from (11) and (13) in an *a priori* manner using the DNS data to compute S_k, α, u_r, and C_D. They are shown in Figure 1, together with the DNS data from [23]. Also shown for comparison are the predictions from the full differential two equation model of [14]. It can be seen that despite their simplicity, the

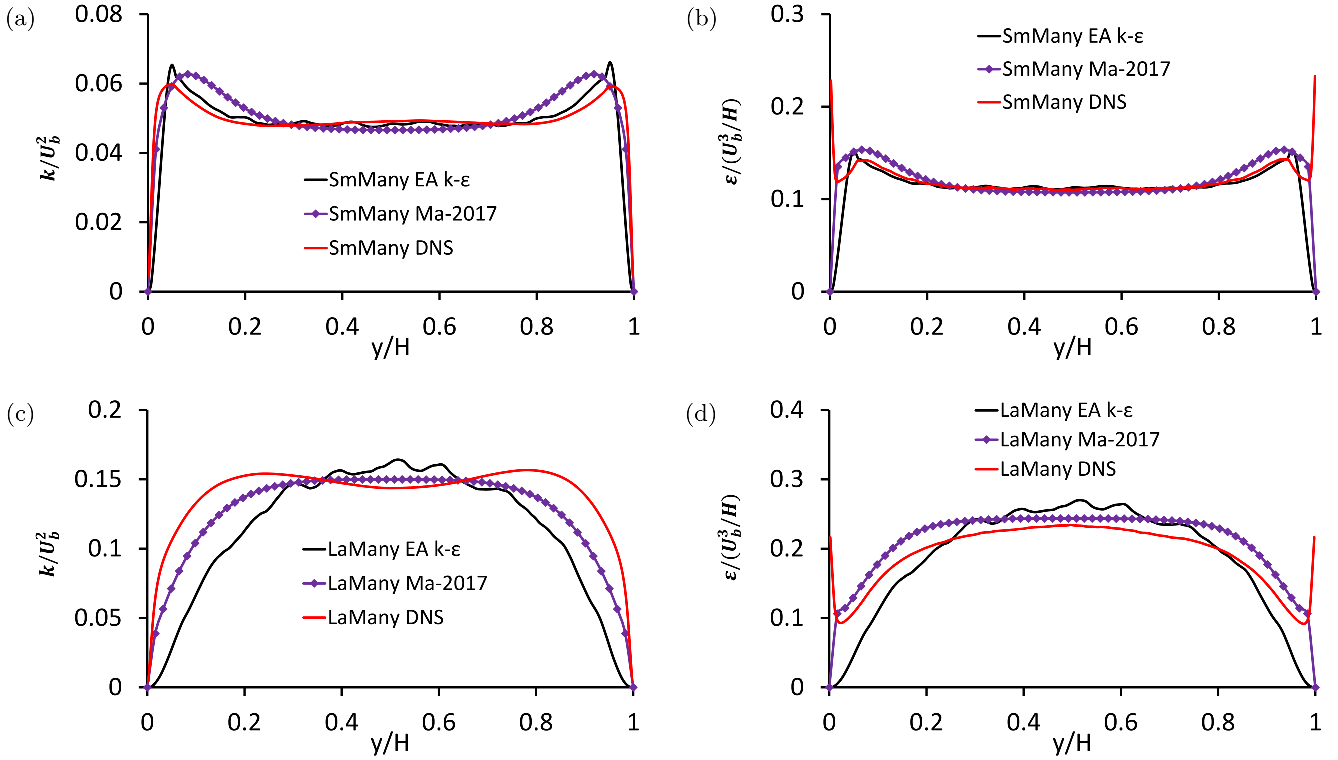

FIG. 1: Liquid TKE and dissipation from the present explicit algebraic (EA) $k-\varepsilon$ expression, two equation model of [14], and DNS data for two cases: (a,b) *SmMany*; and (c,d) *LaMany*.

algebraic expressions (11) and (13) describe the behavior of ε and k well in the channel center of the *SmMany* case for which they were derived. For the *LaMany* case the agreement is not quite as good as that obtained with the full differential two equation model of [14].

We now test (10) by comparing its predictions for the Reynolds normal stresses in three test cases against the DRSM of [8], as well as DNS and experimental data. It should be noted that in the DRSM of [8], the non-linear pressure-strain model of [25] is used (see (B1) in Appendix B), which is different compared to the linearized version of the model stated in (2). Figure 2 shows results obtained with (10) for the bubbly channel (again, *SmMany*, *LaMany* cases) and the bubble column, labeled *Akbar3*. The *Akbar* case [26] features a rectangular water/air bubble column, with a gas superficial velocity of 3 mm/s. Furthermore, this case is very close to monodisperse with $d_p = 4.37\ mm$ and $Re_p \approx 1080$. The coordinate system of the DNS cases is used for improved readability, which is different from the one employed in the original paper. More details are provided in [26] and [27].

When computing $\overline{u'u'}$ and $\overline{v'v'}$ using (10), the required quantities ε and k are obtained using (11) and (13), respectively, and the interfacial term $S_{R,ij}$ is obtained from (3), with α, d_p, u_r, and C_D from either DNS or experiment. Moreover, S_k in (3) is determined from DNS for the *SmMany* and *LaMany* cases [23], while for the case *Akbar3*, S_k is specified using the model of [14].

Two important observations can be made concerning the results in Figure 2. First, the streamwise component of the Reynolds normal stress is predicted more accurately than the wall-normal direction (the absolute errors of the predictions for the streamwise and wall-normal components are similar, however, when considering the relative error, the streamwise component is seen to be the most accurately predicted), especially in the cases *SmMany* and *LaMany*. Second, the prediction is improved and approaches the quality of DRSM for both $\overline{u'u'}$ and $\overline{v'v'}$ in the sequence *SmMany*, *LaMany* to *Akbar3*, which corresponds to increasing bubble Reynolds number, Re_p. The main reason the model predicts $\overline{v'v'}$ less accurately than $\overline{u'u'}$ is associated with the sensitivity of $\overline{v'v'}$ to the pressure-strain term. This is confirmed by Figure 7 in [8] which shows that in contrast to streamwise component the pressure-strain term is the dominated source term contributing in the Reynolds stress budget of the wall-normal direction, rather than the interfacial term. The algebraic expression (10) is obtained using the linear model of Rotta for the pressure-strain term, which can lead to inaccuracies, whereas [8] utilizes the superior non-linear model of [25] and so leads to better

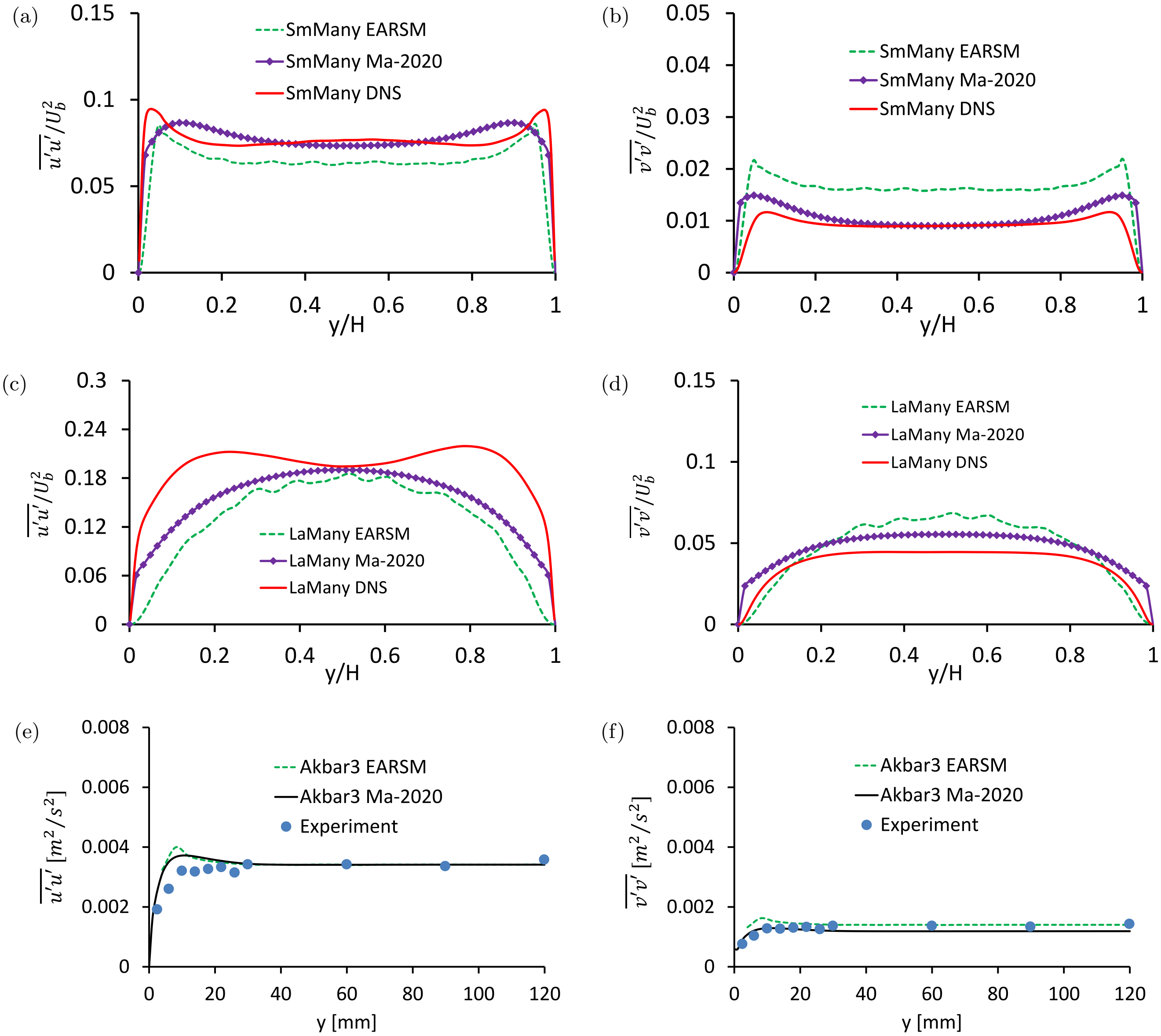

FIG. 2: Liquid Reynolds-stress components from the present EARSM, DRSM of [8], and DNS/experimental data for three test cases: (a,b) *SmMany*; (c,d) *LaMany*; and (e,f) *Akbar3*.

predictions for $\overline{v'v'}$. As Re_p increases, the role of the pressure-strain term weakens relative to the contribution from the interfacial term. This is why the predictions in the channel center from the new EARSM approach those of the DRSM of [8] in the sequence *SmMany*, *LaMany* to *Akbar3*, since the errors associated with the linear modeling of the pressure-strain term become less important as Re_p increases.

IV. CONCLUSIONS

Two new explicit algebraic turbulence models have been developed for flows dominated by BIT. The first model is an explicit algebraic model for the Reynolds stress based on the linearised DRSM of [8]. This model requires the information about k and ε as input, which could be obtained from a two equation turbulence model. However, we have derived a second explicit algebraic model for k and ε. When these two models are used together, it yields to a purely algebraic model that is able to predict the Reynolds normal stresses in BIT dominated flows.

We have demonstrated that these models can predict the Reynolds normal stresses well, especially for larger bubble Reynolds numbers, and the results achieved are comparable to those of the full differential RSM. The simplicity of these purely algebraic models makes them very attractive for use in engineering calculations to efficiently predict bubbly turbulent flows, and may be useful in the development of future BIT models.

Despite the success of these models, they, like all models, have their limitations. In particular the models apply to the core of the wall bounded flows, but not the near wall-regions, and is less accurate for smaller bubble Reynolds numbers. Moreover, it should be noted that the explicit algebraic Reynolds stress model we have developed is not a complete Reynolds stress closure, since the model is unable to provide the information of off-diagonal components for which its prediction is

$$\overline{u'v'} = \frac{k}{c_1(1-\alpha)\varepsilon}\underbrace{b^*_{12}}_{=0} S_k + \frac{2}{3}\underbrace{\delta_{12}}_{=0} k\left(1-\frac{1}{c_1}\right) = 0\,. \tag{14}$$

This deficiency can be traced back to the original DRSM of [8], upon which our algebraic model is based. In this DRSM, the interfacial term is modeled as a diagonal matrix (3), so that the off-diagonal terms that in reality contribute to $\overline{u'v'}$ are absent. In future work we will seek to further develop the models to address these limitations.

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Appendix A: Form of the modelled interfacial term in two-equation Euler-Euler RANS given by [14] and the governing equations of the Euler-Euler approach

The model equations of [14] for BIT flows are written within the Euler-Euler $k-\varepsilon$ framework, and have the following form

$$\frac{D((1-\alpha)k)}{Dt} = P_k + D_k \underbrace{-(1-\alpha)\varepsilon}_{\varepsilon_k} + \underbrace{\min(0.18\cdot Re_p^{0.23},1)\boldsymbol{F}_D\cdot(\boldsymbol{u}^G-\boldsymbol{u}^L)}_{S_k}, \tag{A1}$$

$$\frac{D((1-\alpha)\varepsilon)}{Dt} = P_\varepsilon + D_\varepsilon \underbrace{-(1-\alpha)C_{\varepsilon 2}\frac{\varepsilon^2}{k}}_{\varepsilon_\varepsilon} + \underbrace{0.3\cdot C_D\frac{S_k}{\tau}}_{S_\varepsilon}\,,\ with\, \tau = \frac{d_p}{u_r}. \tag{A2}$$

Away from the walls, the dominant terms are the interfacial terms (S_k and S_ε) and dissipation terms (ε_k and ε_ε), where $C_{\varepsilon 2} = 1.92$. P_ε and D_ε are the production and diffusion terms, respectively, in the modelled ε-equation.

For an incompressible gas-liquid, two-phase flow without phase transition, the governing equations within the Euler-Euler framework [1] are

$$\frac{\partial(\alpha^K\rho^K)}{\partial t} + \nabla\cdot(\alpha^K\rho^K\boldsymbol{u}^K) = 0, \tag{A3}$$

$$\frac{D(\alpha^K\rho^K\boldsymbol{u}^K)}{Dt} = \nabla\cdot(2\alpha^K\mu^K\boldsymbol{S}^K) - \alpha^K\nabla p + \alpha^K\rho^K\boldsymbol{g} + \boldsymbol{M}^K - \nabla\cdot(\alpha^K\boldsymbol{\tau}_t^K), \tag{A4}$$

where all quantities are mean values. The superscript K denotes the phases (L liquid, G gas), with μ, $\boldsymbol{u}$ and $\boldsymbol{S}$ being the molecular viscosity, the mean velocity, and the mean strain rate tensor, respectively. The unresolved stress tensor $\boldsymbol{\tau}_t$ and the sum of all interfacial forces $\boldsymbol{M}$ acting on phase K have to be modeled.

Appendix B: The pressure-strain model used in [8]

The modeled pressure-strain term ϕ_{ij} used in [8] is identical to the SSG model [25] (the two-phase version), and is given by

$$\begin{aligned}\phi_{ij} =& -c_1(1-\alpha)\varepsilon a_{ij} + c_1'(1-\alpha)\varepsilon\Big(a_{ik}a_{kj} - \frac{1}{3}\delta_{ij}A_2\Big) \\ & - c_2^*\Big(P_{ij} - \frac{1}{3}\delta_{ij}P_{kk}\Big) - c_3^*\Big(E_{ij} - \frac{1}{3}\delta_{ij}E_{kk}\Big) - c_4^*(1-\alpha)kF_{ij} - c_5^* a_{ij}P_{kk}\ ,\end{aligned} \tag{B1}$$

where P_{ij} is the production, and

$$E_{ij} = -(1-\alpha)\Big(\overline{u_i'u_k'}\frac{\partial \overline{u_k}}{\partial x_j} + \overline{u_j'u_k'}\frac{\partial \overline{u_k}}{\partial x_i}\Big), \quad F_{ij} = \frac{1}{2}\Big(\frac{\partial \overline{u_i}}{\partial x_j} + \frac{\partial \overline{u_j}}{\partial x_i}\Big)\ . \tag{B2}$$

The constants in (B1) are [15]

$$\begin{gathered} c_1 = 1.7, \quad c_1' = 1.05, \quad c_2^* = 0.4125, \quad c_3^* = 0.2125, \\ c_4^* = 0.033 + 0.65A_2^{1/2}, \quad c_5^* = 0.45\ . \end{gathered}$$

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